Analytical approach to critical collapse in 2+1 dimensions[1]

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Abstract

We present a family of time-dependent solutions to 2+1 gravity with negative cosmological constant and a massless scalar field as source. These solutions are continuously self-similar near the central singularity. We analyze linear perturbations of these solutions, and discuss the subtle question of boundary conditions. We find two growing modes, one of which corresponds to the linearization of static singular solutions, while the other describes black hole formation.

1 Introduction

In recent numerical simulations [1] of gravitational collapse of a massless scalar field in 2+1 dimensions with negative cosmological constant Λ, threshold solutions were observed, exhibiting features characteristic of critical collapse [2], namely power-law scaling and continuous self-similarity (CSS). It has been argued by Garfinkle [3] that this behavior is governed by an exact CSS solution of the $\Lambda = 0$ theory. To show that such a solution is indeed a critical solution, one should investigate near-critical configurations to see whether they describe black hole formation. However, as a strictly negative cosmological constant is necessary for the existence of black holes in 2+1 dimensions [4], one should first extend the Garfinkle solutions to $\Lambda < 0$, a hard problem which has not yet been solved in generality.

In this talk, we shall present a new class of $\Lambda = 0$ CSS solutions which can be extended to $\Lambda < 0$, and show that a class of nearby solutions do

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indeed describe black hole formation. After reviewing the Garfinkle solutions, we describe the new CSS solutions and their extension to $\Lambda \neq 0$ quasi-CSS solutions. We then perform the linear perturbation analysis in the background of these threshold solutions. Imposing appropriate boundary conditions, we obtain discrete modes describing black hole formation, and discuss the determination of the critical exponent.

2 CSS and quasi-CSS solutions

The Einstein-massless scalar field equations with cosmological constant are

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - (1/2) g_{\mu\nu} \partial^\lambda \phi \partial_\lambda \phi, \qquad \nabla^2 \phi = 0. \tag{1}$$

Assuming rotational symmetry, we make the double null ansatz:

$$ds^2 = e^{2\sigma(u,v)} du dv - r^2(u,v) d\theta^2, \quad \phi = \phi(u,v). \tag{2}$$

The field equations then take the form

$$r_{,uv} = (\Lambda/2) r e^{2\sigma}, \tag{3}$$
$$2\sigma_{,uv} = (\Lambda/2) e^{2\sigma} - \phi_{,u}\phi_{,v}, \tag{4}$$
$$2\sigma_{,u} r_{,u} - r_{,uu} = r\phi_{,u}^2, \tag{5}$$
$$2\sigma_{,v} r_{,v} - r_{,vv} = r\phi_{,v}^2, \tag{6}$$
$$2r\phi_{,uv} + r_{,u}\phi_{,v} + r_{,v}\phi_{,u} = 0 \quad . \tag{7}$$

Assuming $\Lambda = 0$, Garfinkle found [3] the following exact CSS solutions

$$ds^2 = -A \left(\frac{(\sqrt{\hat{v}} + \sqrt{-\hat{u}})^4}{-\hat{u}\hat{v}} \right)^{c^2} d\hat{u}\, d\hat{v} - (\hat{v} + \hat{u})^2\, d\theta^2 \,, \tag{8}$$
$$\phi = -2c \ln(\sqrt{\hat{v}} + \sqrt{-\hat{u}}) \,, \tag{9}$$

depending on an arbitrary constant c and a scale $A > 0$. These solutions are continuously self-similar with homothetic vector $(\hat{u}\partial_{\hat{u}} + \hat{v}\partial_{\hat{v}})$ and, for $c \simeq 1$, agree well near the singularity with numerical results of [1]. However, it has not been possible up to now to extend them to solutions of the full $\Lambda < 0$ equations with both CSS behaviour near the singularity, and the correct AdS behaviour at spatial infinity.

From the Garfinkle class of CSS solutions (8)-(9), we derive a new class of CSS solutions by an infinite boost $\hat{u} \to \lambda^{-1}\hat{u}$, $\hat{v} \to \lambda \hat{v}$ ($\lambda \to 0$), arriving at

$$ds^2 = -\bar{A}(-\hat{u}/\hat{v})^{c^2} d\hat{u} d\hat{v} - \hat{u}^2 d\theta^2. \tag{10}$$

These new solutions obviously inherit the CSS behaviour of the original Garfinkle solutions. The transformation to new null coordinates $u = -(-\hat{u})^{1+c^2}$, $v = \hat{v}^{1-c^2}$ leads to the simpler form of these solutions

$$ds^2 = dudv - (-u)^{2\alpha} d\theta^2, \quad \phi = -c\alpha \ln(-u) \tag{11}$$

($\alpha = 1/(1+c^2)$), which may be checked to solve exactly Eqs. (3)-(7).

The generic solution (11) has four Killing vectors. While this spacetime is devoid of scalar curvature singularities, the study of geodesic motion shows that for $c^2 < 1$ only radial geodesics can be continued through the null line $u = 0$, while nonradial geodesics terminate at the singular point $u = 0, v \to +\infty$. On the other hand, for $c^2 > 1$ nonradial geodesics terminate on the null line $u = 0$. So the $c^2 = 1$ solution is an extreme solution separating the $c^2 \leq 1$ solutions with a point singularity from the $c^2 > 1$ solutions with a null line singularity.

Now we proceed to extend the second class of $\Lambda = 0$ CSS solutions (11) to exact quasi-CSS solutions of the full $\Lambda < 0$ equations (the self-similarity being then broken by the cosmological constant). We make the ansatz

$$ds^2 = e^{2\nu(x)} dudv - (-u)^{2\alpha} \rho^2(x) d\theta^2, \quad \phi = -c\alpha \ln(-u) + \psi(x) \quad (x \equiv uv). \tag{12}$$

Inserting this ansatz into the field equations (3)-(7) leads to the system

$$\begin{aligned} x\rho'' + (1+\alpha)\rho' &= (\Lambda/2)\rho e^{2\nu}, & (13)\\ 2(x\nu'' + \nu') + \psi'(x\psi' - c\alpha) &= (\Lambda/2)e^{2\nu}, & (14)\\ x^2(-\rho'' + 2\rho'\nu' - \rho\psi'^2) + 2\alpha x(-\rho' + \rho(\nu' + c\psi')) &= 0 & (15)\\ -\rho'' + 2\rho'\nu' - \rho\psi'^2 &= 0 & (16)\\ 2x(\rho\psi')' + (2+\alpha)\rho\psi' &= c\alpha\rho'. & (17) \end{aligned}$$

A simple first integral is

$$\rho = e^{\nu + c\psi}. \tag{18}$$

The numerical solution of this system with the boundary conditions

$$\rho(0) = 1, \quad \nu(0) = 0, \quad \psi(0) = 0 \tag{19}$$

leads [5] to a unique extension which reduces to (11) near the singularity $u = 0$ and which, for $x \to x_1 < 0$, is asymptotic to the AdS_3 (anti-de-Sitter) spacetime

$$ds^2 = (X^2+1)dT^2 - \frac{dX^2}{X^2+1} - X^2 d\theta^2. \tag{20}$$

3 Perturbations

To ascertain whether the quasi-CSS solutions of the preceding section are threshold solutions for black hole formation, we study their linear perturbations, along the lines of the analysis of [6]. The relevant time parameter in critical collapse being the retarded time [3] $T = -\ln(-U) = -\alpha\ln(-u)$, we expand these perturbations in modes proportional to $e^{kT} = (-u)^{-k\alpha}$. Keeping only one mode, we decompose the perturbed fields as

$$\begin{aligned} r &= (-u)^{\alpha}[\rho(x) + \lambda(-u)^{-k\alpha}\tilde{r}(x)], & (21)\\ \sigma &= \nu(x) + \lambda(-u)^{-k\alpha}\tilde{\sigma}(x), & (22)\\ \phi &= -c\alpha\ln(-u) + \psi(x) + \lambda(-u)^{-k\alpha}\tilde{\phi}(x). & (23)\end{aligned}$$

The linearization of the Einstein equations (3)-(7) in the small parameter λ leads to the fourth order system for the perturbations $\tilde{r}$, $\tilde{\sigma}$, $\tilde{\phi}$:

$$\begin{aligned} & x\tilde{r}'' + (1+\alpha-k\alpha)\tilde{r}' = (\Lambda/2)\,e^{2\nu}(\tilde{r} + 2\rho\tilde{\sigma}), & (24)\\ & 2x\tilde{\sigma}'' + 2(1-k\alpha)\tilde{\sigma}' = \Lambda e^{2\nu}\tilde{\sigma} - (2x\psi' - c\alpha)\tilde{\phi}' + k\alpha\psi'\tilde{\phi}, & (25)\\ & -(1-k)x\tilde{r}' + ((1-k)x\nu' + (k/2)(2\alpha - 1 - k\alpha))\tilde{r} + \rho x\tilde{\sigma}' - k(x\rho' + \alpha\rho)\tilde{\sigma} = \\ & \qquad -\rho(cx\tilde{\phi}' - k(c\alpha - x\psi')\tilde{\phi}) - cx\psi'\tilde{r}, & (26)\\ & 2(\rho'\tilde{\sigma}' + \nu'\tilde{r}') - \tilde{r}'' = \psi'(2\rho\tilde{\phi}' + \psi'\tilde{r}), & (27)\\ & 2x\rho\tilde{\phi}'' + (2x\rho' + (2+\alpha-2k\alpha\rho)\tilde{\phi}' - k\alpha\rho'\tilde{\phi} + (2x\psi' - c\alpha)\tilde{r}' \\ & \qquad +(2x\psi'' + (2+\alpha-k\alpha)\psi')\tilde{r} = 0. & (28)\end{aligned}$$

This system admits the exact, spurious solutions (gauge modes)

$$\begin{aligned} \tilde{r}_k^{(1)} = (-x)^{p_+}\rho',\ \tilde{\sigma}_k^{(1)} = (-x)^{p_+}\nu' - \tfrac{p_+}{2}(-x)^{p_+-1},\ \tilde{\phi}_k^{(1)} = (-x)^{p_+}\psi', \quad (29)\\ \tilde{r}_k^{(2)} = x\rho'(x) + \alpha\rho,\ \tilde{\sigma}_k^{(2)} = x\nu' + \tfrac{p_-}{2},\ \tilde{\phi}_k^{(2)} = x\psi' - c\alpha, \quad (30)\end{aligned}$$

($p_\pm = 1 \pm k\alpha$) generated respectively by the small gauge transformations $v \to v - \lambda v^{p_+}$ and $u \to u - \lambda(-u)^{p_-}$. So, up to gauge transformations, the general solution of this system depends only on two integration constants, which should be determined by enforcing appropriate boundary conditions.

The three boundaries of our quasi-CSS solutions are $u = 0$, $v = 0$ (or $x = 0$), and $x = x_1$ (the AdS boundary). On the singular boundary $u = 0$, it seems natural to impose that the perturbed metric component r does not diverge too quickly, which would conflict with the linear approximation. However we shall argue that it may be preferable to replace this condition (which is subject to some ambiguity) with a boundary condition on the

apparent horizon. On the second boundary $v = 0$, we shall require that the perturbed solution matches smoothly the original quasi-CSS solution

$$\tilde{r}(0) = 0, \quad \tilde{\sigma}(0) = 0, \quad \tilde{\phi}(0) = 0, \tag{31}$$

and moreover that the perturbations be (as the unperturbed quasi-CSS solution) analytic in v. This last condition ensures regularity on the line $v = 0$. Finally, the perturbations should grow slowly (i.e. not faster than the original fields) as the AdS boundary $x = x_1$ is approached. It turns out that this last condition is always satisfied up to gauge transformations. This is due to the fact that the perturbed scalar field becomes negligible at spatial infinity, so that the asymptotic behavior is that of AdS_3 (the effect of gauge transformations is to shift the value x_1 for which the AdS boundary is attained).

To enforce our boundary conditions on the line $v = 0$, it is enough to know the behavior of the solutions of the linearized system near $x = 0$. Up to gauge transformations, this is given in terms of two integration constants B, C by

$$\tilde{r}(x) \simeq B(-x) + \Lambda C(-x)^{1+(k-1/2)\alpha}, \tag{32}$$

$$\begin{aligned}\tilde{\sigma}(x) \simeq -\tilde{\phi}(x) &\simeq -(1+\alpha-k\alpha)\Lambda^{-1}B + O(x) \\ &\quad -(1+\alpha/2)(1+(k-1/2)\alpha)C(-x)^{(k-1/2)\alpha}.\end{aligned} \tag{33}$$

for $k \neq 1/2$ (in the degenerate case $k = 1/2$, the power law $(-x)^{(k-1/2)\alpha}$ is replaced by a logarithm). From these behaviors, we find that the smooth matching conditions at $v = 0$ are satisfied if either

$$C = 0, \quad k = c^2 + 2, \tag{34}$$

(in this case the analyticity condition is also satisfied), or $B = 0$ which, taking into account the analyticity condition, implies

$$B = 0, \quad k = n(c^2 + 1) + 1/2, \tag{35}$$

where n is a positive integer. In this last case the boundary condition that the perturbed r does not diverge too fast (that is, no faster than in the BTZ case, see Sect. 4) when $u \to 0$ with x fixed selects the eigenvalue $n = 1$, i.e. $k = c^2 + 3/2$.

4 Black hole formation

Black hole formation is characterized by the appearance of a central singularity hidden by an apparent horizon. The definition of these notions can be

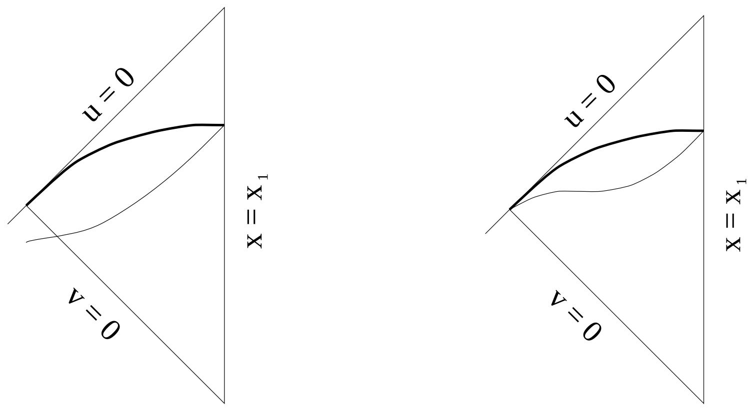

Figure 1: Perturbative singularity (thick curve) and apparent horizon for solutions (34) (left) and (35) (right)

quite ambiguous in linearized perturbation theory. For our present purpose we shall identify the central singularity with the coordinate singularity of the linearized perturbed metric

$$\sqrt{g} = e^{2\sigma} r = (-u)^{\alpha} e^{2\nu(x)} [\rho(x) + (-u)^{-k\alpha} (\tilde{r}(x) + 2\rho(x)\tilde{\sigma}(x))] = 0, \quad (36)$$

and define the apparent horizon by

$$r_{,v} = (-u)^{1+\alpha} (\rho' + (-u)^{-k\alpha} \tilde{r}') = 0. \quad (37)$$

In the case of the solution (34), for $\lambda B < 0$, a spacelike coordinate singularity appears at $v = 0$, hidden behind a preexisting spacelike apparent horizon, which seems to be eternal (Fig. 1). Such a situation does not seem to correspond to collapse, but rather to a static configuration. Indeed, consider the exact static solution of the Einstein equations (3)-(7) with $\Lambda = 0$ [7]

$$ds^2 = A r^{c^2} (dt^2 - dr^2) - r^2 d\theta^2, \qquad \phi = c \ln r. \quad (38)$$

Putting $r = (-u)^{\alpha} + \lambda v$, $t = -(-u)^{\alpha} + \lambda v$, and expanding in powers of the small parameter λ, we get to first order

$$ds_0^2 = \bar{A}(1 - \lambda c^2 (-u)^{-\alpha-1}) du dv - (-u)^{2\alpha} (1 - \lambda c^2 (-u)^{-\alpha-1})^2 d\theta^2, \quad (39)$$
$$\phi = -c \ln(-u) + \lambda c (-u)^{-\alpha-1} x, \quad (40)$$

which agrees with the first order expansions (21)-(23) for $\Lambda = 0$ and $k\alpha = \alpha + 1$, i.e. $k = c^2 + 2$. We conjecture that the full $\Lambda < 0$ perturbed solution

with $k = c^2 + 2$ results from a similar expansion of the "non-topological soliton" singular static solutions (generalizing (38) to $\Lambda < 0$) [8]

$$\begin{aligned} ds^2 &= A\,|\rho-\rho_+|^{1/2+a}\,|\rho-\rho_-|^{1/2-a}\,dt^2 - \frac{4|\Lambda|}{A}\,|\rho-\rho_+|^{1/2-a}\,|\rho-\rho_-|^{1/2+a}\,d\theta^2 \\ &\quad + \frac{d\rho^2}{4\Lambda(\rho-\rho_+)(\rho-\rho-)}\,, \qquad \phi = \sqrt{\frac{1-4a^2}{8}}\,\ln\left(\frac{|\rho-\rho_+|}{|\rho-\rho_-|}\right) \end{aligned} \tag{41}$$

(with $2a = (c^2-2)/(c^2+2)$). This conjecture is certainly true in the sourceless case $c = 0$, for which these solutions reduce to the BTZ black holes

$$ds^2 = (r^2/l^2 - M)dt^2 - \frac{dr^2}{(r^2/l^2 - M)} - r^2 d\theta^2, \qquad \phi = 0 \tag{42}$$

($l^2 = -\Lambda^{-1}$). The BTZ metric can be rewritten in the double-null form [5]

$$ds^2 = \frac{1}{(1+uv/4l^2)^2}[dudv - (u + (M/4)v)^2 d\theta^2], \tag{43}$$

which corresponds to the first order expansion (21)-(23) for $\rho = e^\nu = (1 + uv/4l^2)^{-1}$, $\phi = 0$, $\lambda = M/4$, $k = 2$ (the linearized solution is exact in this case).

Consider now the second solution (34). For $C > 0$, a coordinate singularity and an apparent horizon, both spacelike, appear simultaneously at $v = 0$, and converge towards the AdS boundary $x = x_1$ for a finite value r_{AH} of the horizon radius (Fig. 1). The fact that this apparent horizon is born together with the singularity suggests that it is not automatically regular. Indeed, the curvature invariant

$$R + 6\Lambda = 4e^{-2\sigma}\phi_{,u}\phi_{,v}, \tag{44}$$

evaluated on the apparent horizon, generically diverges on the original null singularity $u = 0$ because $\phi_{,v}|_{AH} \propto x^{-1} (x \to 0)$. So the apparent horizon can be regular at birth only if

$$e^{-2\sigma}\phi_{,u}|_{AH}(x = 0) = 0, \tag{45}$$

which leads to an eigenvalue equation relating n and c^2. While there is some ambiguity in the linearization of (45), it seems to lead [5] to a single solution $n = 1$, $c^2 \simeq 1$.

Near-critical collapse is characterized by a critical exponent γ, defined by the scaling relation

$$Q \propto |p - p^*|^{s\gamma} \tag{46}$$

for a quantity Q with dimension s depending on a parameter p (with $p = p^*$ for the critical solution). In the perturbative approach, the choice $Q = r_{AH}$ (the apparent horizon radius, $s = 1$) and $p - p^* = \lambda$ (the perturbation amplitude) leads to $\gamma = 1/k$ [6]. For the static BTZ black hole (solution (34) with $c^2 = 0$), we thus obtain

$$\gamma_{(BTZ)} = 1/2, \tag{47}$$

in agreement with previous analyses. In the case of genuine scalar field collapse (solution (35) with $n = 1$) we obtain for the preferred value $c^2 = 1$ the critical exponent

$$\gamma = 2/5. \tag{48}$$

This differs from the two conflicting values ($\gamma \sim 1.2$ and $\gamma \sim 0.8$) found in the numerical analyses of [1], but is of the same order of magnitude.

5 Conclusion

We have constructed a one-parameter (c^2) family of exact time-dependent solutions to the Einstein equations with a negative cosmological constant and a massless scalar field as source. These solutions are continuously self-similar near the singularity, and asymptotically AdS at spatial infinity. We have then analyzed the linear perturbations of these solutions, and discussed the subtle question of boundary conditions. These led to two possible growing modes. The first mode, (34), is characterized by the presence of a presumably eternal apparent horizon. This is simply the linearization of the static BTZ black hole in the sourceless case, and very probably of static singular solutions in the general coupled case. The second mode, (35), exhibits the characteristic features of black hole formation, i.e. the simultaneous appearance of a spacelike singularity and its shielding apparent horizon, which then expands up to a finite size. The condition of regularity at birth of this apparent horizon selects a unique value of the parameter c^2, leading to a value of the critical exponent which is of the order of the values derived from the numerical simulations. While these results are rather satisfactory, we think that further analytical work on this simple 2+1 dimensional model should further clarify the situation and help us to understand better the phenomenon of black hole formation.

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